Journal of Statistical Physics, Vol. 92, Nos. 1/2, 1998

On the Large-Distance Behavior of Correlations for a Hierarchical N-Component Classical Vector Model in Three Dimensions

P. A. Faria da Veiga,[1] M. O'Carroll,[2] and R. Schor[2]

Received February 19, 1998

We consider the correlation functions for a hierarchical N-component classical vector model in three dimensions. For $N = \infty$, we find explicitly the eigenvalues and global eigenfunctions of the linearized renormalization group transformation. In a very direct way, this yields the correlation functions for the $N = \infty$ model. In particular, we check that the two-point function has canonical decay.

KEY WORDS: Large-N vector model; correlation functions; renormalization group.

1. INTRODUCTION

In [M,Z], the classical N-component vector model in three dimensions, $N \gg 1$, is reported to have a nonzero fixed point solution to a renormalization group transformation (RGT) and, at this fixed point, the correlation functions for the model have noncanonical (anomalous) falloff at large distances. A hierarchical version of this model is also studied in [M] and, for $N = \infty$, an explicit nonzero fixed point solution is constructed for a RGT in scaled variables.

Applying rigorous renormalization group (RG) techniques, the same model is studied in [GK1], where a nonzero fixed point is constructed for finite $N \gg 1$. Also a nice and improved review is given of the nonzero fixed point when $N = \infty$, corresponding to the leading order of the $1/N$ expansion for the model, when N is large.

[1] Departamento de Matemática, ICMSC, Universidade de São Paulo, 13560 São Carlos SP, Brazil; e-mail: veiga@icmc.sc.usp.br

[2] Departamento de Física, ICEX, Universidade Federal de Minas Gerais, 31270 Belo Horizonte MG, Brazil; e-mail: ocarroll@oraculo.lcc.ufmg.br

0022-4715/98/0700-0047$15.00/0

In a subsequent paper [GK2] when $N \gg 1$, the techniques of [GK1, GK3, GK4] were also used to determine the large distance behavior of the correlation functions at the corresponding nonzero fixed point. Specifically, it has been shown that the falloff of the two-point function is canonical, i.e. with a *zero* anomalous dimension. Also, it was argued that the correlation functions do satisfy Wick's theorem so that, although nonzero, the constructed fixed point is Gaussian.

Another hierarchical version of the same model has been recently considered in [FOS]. Contrarily to the aforementioned case, where the implemented renormalization group transformation (RGT) is a generalization of the Wilson's transformation [*M*, GK1, GK3) symmetric under the global reflections of the fluctuation fields ($\vec{\eta} \rightarrow -\vec{\eta}$), the RGT is there *nonsymmetric*. Also, the inverse of the associated hierarchical Laplacian matrix, defining the free Gaussian measure for the problem, has positive entries in the nonsymmetric case, as for the full model. Besides, and more interesting for our main concern, it has been shown [FOS], for the very simple example of a scalar ($N=1$) quadratic perturbation, in dimension $d>2$, that an anomalous decay does appear for the correlation functions of the nonsymmetric scalar hierarchical model. The appearance of an anomalous dimension was also seen for the nonsymmetrical hierarchical classical vector model in dimension $d=3$ or higher and the Goldstone picture was confirmed in [SO1, SO2], by showing the change from canonical to noncanonical decay of the two-point function, depending on the considered field direction.

As the main result of [FOS], absence of anomalous decay for the two-point function was shown to take place in the three dimensional nonsymmetric hierarchical classical vector model, up to the second order in the $1/N$ expansion. In principle for large N, there should be no difficulties in applying the techniques and methods of [GK1, GK3, GK4] to extend this results nonperturbatively.

In this notice, we go back to the N-component symmetric model treated in [GK1]. For $N=\infty$, we solve the spectrum of the linearized renormalization group transformation explicitly. The eigenvalues and the global eigenfunctions are used to obtain in a very direct way, the correlation functions for the model. In particular, at this regime, we check that the two-point function has canonical decay in agreement with [GK2].

Albeit the later is a known result, the new proof given below helps to further our understanding of the $N=\infty$ symmetric hierarchical model. Besides, the method developed here can be viewed as an additional step to attack the full model, as it is still to be understood from the point of view of configurations in a real space renormalization group approach and a multiscale Laplace expansion method.

2. THE MODEL AND THE MAIN RESULTS

Our notation is that of [FOS]. Throughout this section most of the formulas hold for generic euclidean dimension d, but our results are established for $d=3$.

For $\ell \in \mathbb{N}$, let Λ_0 denote the lattice $[-L^\ell/2, L^\ell/2)^d \cap \mathbb{Z}^d$, L even, and $\vec{\phi}(x) \in \mathbb{R}^N$, $\vec{\phi}(x) = (\phi_1, (x), ..., \phi_N(x))$, the vector field at $x \in \Lambda_0$. The finite-volume partition function $Z_0 \equiv Z_0(\ell)$ is defined as

$$Z_0 = \int e^{-V(\vec{\phi})}\, d\mu_0(\vec{\phi}) \tag{2.1}$$

where $V(\vec{\phi}) = \sum_{x \in \Lambda_0} v(\vec{\phi}(x))$ and $v(\vec{\phi}(x))$ is assumed to be rotational invariant. $d\mu_0$ is a normalized and mean-zero Gaussian measure. To define its covariance G_0 and with a view to our renormalization group approach, we take $d\mu_0$ as the first element of a sequence of Gaussian measures $d\mu_j$, with covariances G_j, $j=0, 1, ..., \ell$, which we will now define. First. we introduce the lattices $\Lambda_j = [-L^{\ell-j}/2, L^{\ell-j}/2)^d \cap \mathbb{Z}^d$, L even, and let $x^{(j)}$ denote a point of Λ_j. Let B_0 denote the block consisting of the L^d points u, $u_\alpha \in [0, L) \cap \mathbb{Z}$. The sequence of lattices $\{\Lambda_j, j=0, 1, ..., \ell\}$ has the property that $x^{(j)}$ has the unique decomposition $x^{(j)} = Lx^{(j+1)} + u$, $u \in B_0$. We introduce the linear operator $A_j: \ell_2(\Lambda_{(j+1)}) \to \ell_2(\Lambda_j)$ with matrix elements $A_j(x^{(j)}, x^{(j+1)})$ being $+1$ for half of the u's $\in B_0$, -1 for the other half, and zero otherwise.

The point of the above definitions is that the renormalization group gives equal but opposite weights to the fluctuation fields in a L^d block. The mean zero and normalized Gaussian measure $d\mu_j$ has covariance G_j which is diagonal in the component indices and is determined recursively by

$$\begin{aligned} G_j(Lx^{(j+1)} + u, Ly^{(j+1)} + \bar{u}) &= L^{-(d-2)} G_{j+1}(x^{(j+1)}, y^{(j+1)}) \\ &\quad + (A_j \delta_{j+1} A_j^*)(Lx^{(j+1)} + u, Ly^{(j+1)} + \bar{u}) \end{aligned} \tag{2.2}$$

with $u, \bar{u} \in B_0$, $G_\ell = (1 - L^{-(d-2)})^{-1}$ and where δ_j is the identity operator on the lattice Λ_j. We also introduce the fields $\vec{\phi}^{(j)}(x^{(j)})$ on Λ_j. Corresponding to the decomposition of the covariance, we have a decomposition of the fields. With $x^{(j)} = Lx^{(j+1)} + u$ and letting $\vec{\eta}^{(j)}(x^{(j)})$ denote the fluctuation field, it is given by

$$\vec{\phi}^{(j)}(x^{(j)}) = L^{-(d-2)/2} \vec{\phi}^{(j+1)}(x^{(j+1)}) + A_j(x^{(j)}, x^{(j+1)})\, \vec{\eta}^{(j)}(x^{(j+1)}).$$

For $i_1, \ldots, i_p = 1, \ldots, N$, the finite volume p-point correlation function is

$$\langle \phi_{i_1}(x_1) \cdots \phi_{i_p}(x_p) \rangle_\ell \equiv Z_0^{-1} \int \phi_{i_1}(x_1) \cdots \phi_{i_p}(x_p)\, e^{-V(\vec{\phi})}\, d\mu_0(\vec{\phi})$$

with the partition function $Z_0 \equiv Z_0(\ell)$ as given in (2.1), with initial points $x \equiv x^{(0)}$ and initial fields $\vec{\phi}(x) \equiv \vec{\phi}^{(0)}(x^{(0)})$. Applying the vector version of the fundamental identity of [SO1] to the partition function Z_0, we have the chain of identities

$$Z_0 \equiv \int \prod_{x^{(0)} \in \Lambda_0} e^{-V(\vec{\phi}^{(0)}(x^{(0)}))} d\mu_0 = \int \prod_{x^{(1)} \in \Lambda_1} e^{-RV(\vec{\phi}^{(1)}(x^{(1)}))}\, d\mu_1$$

$$\cdots = \int e^{-R^\ell V(\vec{\phi}^{(\ell)}(x^{(\ell)}))}\, d\mu_\ell \equiv Z_\ell$$

where the RGT is defined by

$$e^{-RW(\vec{\phi})} = \int e^{-L^d/2[\, W(L^{-(d-2)/2}\vec{\phi} + \vec{\eta}) + W(L^{-(d-2)/2}\vec{\phi} - \vec{\eta})] - (1/2)\vec{\eta}^2}\, d\vec{\eta} \tag{2.3}$$

The same fundamental identity is useful to represent the correlation functions in terms of effective spin variables. Let M_W be the linear operator given by

$$M_W f(\vec{\phi}) = \frac{\int f(L^{-(d-2)/2}\vec{\phi} + \vec{\eta})\, e^{-L^d/2[\, W(L^{-(d-2)/2}\vec{\phi} + \vec{\eta}) + W(L^{-(d-2)/2}\vec{\phi} - \vec{\eta})] - (1/2)\vec{\eta}^2}\, d\vec{\eta}}{\int (f = 1)}$$

and let $I_i(\vec{\phi}) = \phi_i$. Then, the iterated use of the fundamental identity allows us to write the finite volume two-point function as $(i, j = 1, \ldots, N)$

$$\begin{aligned}\langle \phi_i(0)\, \phi_j(L^k e_1) \rangle_\ell &= Z_1^{-1} \int M_V I_i(\vec{\phi}^{(1)}(0))\, M_V I_j(\vec{\phi}^{(1)}(L^{k-1} e_1)) \\ &\quad \times \prod_{x^{(1)} \in \Lambda_1} e^{-RV(\vec{\phi}^{(1)}(x^{(1)}))}\, d\mu_1 \\ &= \cdots = Z_\ell^{-1} \int \{ (M_{R^{\ell-1}V} \cdots M_{R^k V})([M_{R^{k-1}V} \cdots M_V I_i] \\ &\quad \times [M_{R^{k-1}V} \cdots M_V I_i]) \}(\vec{\phi}^{(\ell)})\, e^{-R^{\ell-1}V(\vec{\phi}^{(\ell)})}\, d\mu_\ell\end{aligned}$$

Note that, when evaluated at a fixed point V of the RGT, M_V is L^{-d} times the corresponding linearized RGT operator. This observation is helpful for understanding scaling properties of effective spin variables. Similar formulas can be written for other correlation functions. From the above representation at a fixed point V, and for the thermodynamic limit, an infinite number of compositions of the M_V's is needed. Since, as seen below, only the constant functions are marginal, the infinite composition projects onto the constants.

We now specialize to the large N regime where it is convenient to use reduced scaled variables. Using the rotational symmetry of the fixed point, we write

$$\vec{\phi} = N^{1/2}(\varphi, 0, \ldots, 0); \quad \vec{\eta} = N^{1/2}(s, \zeta_1, \ldots, \zeta_{N-1}), \quad \zeta_1^2 + \cdots \zeta_{N-1}^2 = u \tag{2.4}$$

with $-\infty < s < \infty$, $0 \leqslant u < \infty$, such that $d\vec{\eta} = ds\, u^{N/2 - 3/2}\, du\, d\Omega_{N-1}$. Upon substituting $v(\vec{\phi}) \to (N/2)\, v(\varphi^2)$, $Rv(\vec{\phi}) \to (N/2)\, rv(\varphi^2)$, the RGT (2.3) becomes

$$\begin{aligned} e^{-(N/2)\, rv(\varphi^2)} = N\Omega_{N-1} \int_{-\infty}^{\infty} ds \int_0^{\infty} du \\ \times e^{-(N/2)\{L^3/2[v(\psi_+) + v(\psi_-)] + s^2 + u - \ln u\} - 3/2 \ln u} \end{aligned} \tag{2.5}$$

$\psi_\pm \equiv \psi_\pm(s, u) = (L^{-1/2}\varphi \pm s)^2 + u$ and Ω_{N-1} is the $(N-1)$-dimensional solid angle. For $N = \infty$, we take the RGT as

$$\begin{aligned} r(v) &\equiv \inf_{s,u} \left[\frac{L^3}{2} (v(\psi_+) + v(\psi_-)) + s^2 + u - \ln u \right] \\ &= L^3 v(\psi_+) + s_0^2 + u_0 - \ln u_0 \end{aligned} \tag{2.6}$$

where $s_0(\varphi^2)$ and $u_0(\varphi^2)$ are solutions of the equations

$$\begin{aligned} L^3[v'(\psi_0^+)(L^{-1/2}\phi + s_0) + v'(\psi_0^-)(L^{-1/2}\varphi - s_0)] + 2s_0 = 0 \\ \frac{1}{2} L^3[v'(\psi_0^+) + v'(\psi_0^-)] + 1 - \frac{1}{u_0} = 0 \end{aligned} \tag{2.7}$$

where $v' = dv/d\varphi^2$, $\psi_0 \equiv \psi_{\pm,0} = \psi_\pm(s_0, u_0)$. Thus, since $s_0(\varphi^2) = 0$ is a solution

$$r(v) = L^3 v(L^{-1}\varphi^2 + u_0) + u_0 - \ln u_0$$

and upon taking a derivative $rv'(\varphi^2) = L^2 v'(L^{-1}\varphi^2 + u_0)$.

At the $N = \infty$ fixed point v, we set $\tau(\varphi^2) \equiv v'(\varphi^2) = L^2 v'(L^{-1}\varphi^2 + u_0)$, as in [M, GK1], such that v is determined by

$$t(\tau) = \varphi_0^2 + \sum_{n=0}^{\infty} L^{2-n}\tau(1 + L^{1-2n}\tau)^{-1}; \qquad \varphi_0^2 = L/(L-1) \tag{2.8}$$

where t is the inverse function of v' on $(-L^{-1}, \infty)$. v' has a unique minimum at $\varphi^2 = \varphi_0^2$. As $t(\tau)$ admits an analytic extension for each $\tau \in (-L^{-1}, \infty)$ and $t'(\tau) > 0$, for $\tau \in (-L^{-1}, \infty)$, by the analytic inverse function theorem (see [H]) v' exists and is analytic. In fact, in the analytic inverse function theorem, the inverse function can be made explicit (see [SO3]). The Taylor expansion of v around φ_0^2, and normalized by $v(\varphi_0^2) = 0$, is

$$v(\varphi^2) = \frac{1}{2}\mu_\infty(\varphi^2 - \varphi_0^2) + \sum_{n=3}^{\infty} b_n(\varphi^2 - \varphi_0^2)^n; \qquad \mu_\infty = \frac{L-1}{L^3} \tag{2.9}$$

The $N = \infty$ linearized RGT $\mathscr{L}_{rv}$ is given by

$$\mathscr{L}_{rv} f(\varphi^2) = L^3 f(\psi_0 = L^{-1}\varphi^2 + u_0) \tag{2.10}$$

as can be derived directly from the reduced RGT [see (2.5)] by the Laplace method.

Recalling that we set $d = 3$, we now state our results. The operator $\mathscr{L}_{rv}$ has the spectral properties given in:

Theorem 1. Let v be the $N = \infty$ fixed point of [M, GK1]. Then, the operator $\mathscr{L}_{rv}$ has eigenvalues $L^3 L^{-2n}$ with corresponding eigenfunctions $[v'(\varphi^2)]^n$, $n = 0, 1, \ldots$

Theorem 1 gives us insight into the $N = \infty$ model. As near the minimum φ_0^2 of v, $v' \sim (\varphi^2 - \varphi_0^2)$, besides the constants, we see that the linear form $\varphi^2 - \varphi_0^2$ is relevant, i.e., an expanding direction. The directions $(\varphi^2 - \varphi_0^2)^j$, $j > 1$, are irrelevant, i.e., contractive. Considering the correlation functions represented in terms of reduced variables the reduced linearized operator M_V, denoted by m, has eigenvalues L^{-2n}, with $n = 0, 1, \ldots$. Thus, in the thermodynamic limit, only constants (corresponding to $n = 0$) are expected to survive. This phenomenon can be seen in the inductive description of effective spin variables in the hierarchical *grad* ϕ, and ϕ_4^4 models in [GK3].

Concerning the correlation functions, using certain functional relations, we show:

Theorem 2. For $N = \infty$, the thermodynamic limit of the two-function exists, has canonical decay and is given by

$$\langle \phi_i(0)\, \phi_j(L^k e_1) \rangle \equiv \lim_{\ell \to \infty} \langle \phi_i(0)\, \phi_j(L^k e_1) \rangle_\ell = \delta_{ij} \varphi_0^2 \frac{1}{(L^k)^{d-2}}$$

3. PROOFS

Referring to the representation in Section 2, the fluctuation integrals that occur in the evaluation of the two-point function are zero until both effective spin variables are in the block at zero. Using the rotational symmetry we have, with $\vec{I}^2(\vec{\phi})=\vec{\phi}^2$,

$$\langle \phi_i(0)\,\phi_j(L^k e_1)\rangle_\ell = \frac{\delta_{ij}}{N} L^{-k} \int (M_{R^{\ell-1}V}\cdots M_{R^k V}\vec{I}^2)(\vec{\phi}^{(\ell)})\, e^{-R^{\ell-1}V(\vec{\phi}^{(\ell)})}\, d\mu_\ell/Z_\ell \tag{3.1}$$

In terms of the reduced variables (2.4), for $\varphi\equiv\varphi^{(\ell)}$ and with $I(\varphi^2)=\varphi^2$ the two-point function (3.1) can be written as

$$\begin{aligned}
&\langle \phi_1(0)\,\phi_j(L^k e)\rangle_\ell \\
&\quad = \delta_{ij} L^{-k} \int_{-\infty}^{\infty} [(m_{\ell-1}\cdots m_k I)(\varphi^2)] \\
&\qquad \times e^{-(1/2)Nv_{\ell-1}(\varphi^2)} e^{-(1/2)N(1-L^{-1})[\varphi^2]} e^{(N-1)\ln\varphi}\, d\varphi \Big/ \int [=1]\, d\varphi
\end{aligned}$$

where m_j is the linear operator

$$m_j f(\varphi^2) = \frac{\begin{pmatrix} \int_{-\infty}^{\infty} ds \int_0^\infty du\, f((L^{-1/2}\varphi+s)^2+u) \\ \times e^{-N/2\{L^{3/2}[v_j(\psi_+)+v_j(\psi_-)]+s^2+u-\ln u\}-3/2\ln u} \end{pmatrix}}{\int [f\to 1]}$$

with $f\equiv f(\varphi^2)$ and $v_j=r^j v$. At a fixed point, set $m_j=m$ and $v_j=v$. The $d=3$, $N=\infty$ correlation functions can be defined by evaluating all integrals, except for the last, at the fixed point, and the last integral at the minimum of

$$\frac{L^3}{2}[v(\psi_+)+v(\psi_-)]+\frac{1}{2}(u-\ln u)+\frac{1-L^{-(d-2)}}{2}\varphi^2-\ln\varphi$$

The minimum occurs at $\varphi^2=\varphi_0^2$, $s=s_0=0$; $u=u_0(\varphi_0^2)=1$. Thus, we have

$$\langle \phi_i(0)\,\phi_j(L^k e_1)\rangle_\ell = \delta_{ij} L^{-2k}[m^{\ell-k} I](\varphi_0^2) \tag{3.2}$$

where $mf(\varphi^2)=f(L^{-1}\varphi^2+u_0)$.

An important observation is that

$$f(L^{-1}\varphi^2+u_0)=(f\circ\mathscr{G})(\varphi^2) \tag{3.3}$$

where $\mathcal{G}(\varphi^2) = L^{-1}\varphi^2 + u_0$. Hence

$$m^p f = f \circ \overset{1}{\mathcal{G}} \circ \cdots \circ \overset{p}{\mathcal{G}} \tag{3.4}$$

The key functional relations to be used below are the fixed point equations for v', namely

$$v' = L^2 v' \circ \mathcal{G} \tag{3.5}$$

and, applying t to $L^{-2}v'$,

$$\mathcal{G} = t \circ (L^{-2}v') \tag{3.6}$$

Thus, iterating (3.5) gives $v' = L^{2p} \circ \overset{1}{\mathcal{G}} \circ \overset{2}{\mathcal{G}} \circ \cdots \circ \overset{p}{\mathcal{G}}$, and applying t, after multiplication by L^{-2p}, we obtain

$$\overset{1}{\mathcal{G}} \circ \overset{2}{\mathcal{G}} \circ \cdots \circ \overset{p}{\mathcal{G}} = t(L^{-2p}v') \tag{3.7}$$

For the thermodynamic limit, we need $\lim_{p\to\infty}[m^p I = I\overset{1}{\mathcal{G}} \circ \cdots \circ \overset{p}{\mathcal{G}} = t(L^{-2p}v')] = t(0) = \varphi_0^2$. Thus, it follows that $\langle \phi_i(0)\, \phi_j(L^k e)\rangle \equiv \lim_{\ell\to\infty} \langle \phi_i(0)\, \phi_j(L^k e_1)\rangle_\ell = \delta_{ij} L^{-k}\varphi_0^2$.

We now turn to the eigenvalue problem for the linear operator m of (3.4). We show that v'^n is an eigenfunction with eigenvalue L^{-2n}. From (3.6), with $p=1$, and (3.7), we have $mv'^n = v'^n \circ (t(L^{-2}v')) = [v'(t(L^{-2}v'))]^n = L^{-2n}v'^n$. Recalling (2.10) and (3.3), this concludes the proofs of Theorems 1 and 2. ∎

4. FINAL COMMENT

Now that we have the spectral decomposition of m, to end, we reconsider formally the calculation of $m^j f$, as the one of Eq. (3.2), $f = I$, from a spectral point of view. If we can manage to write f in terms of the eigenfunctions $f_n \equiv v'^n$, i.e., $f = \sum_n \beta_n f_n$, then $m^j f = \sum_n \beta_n L^{-2jn} f_n$, and $\lim_{j\to\infty} m^j f = \beta_0$. Since our $f(\varphi^2) = \varphi^2$, we see that the Taylor expansion of $t(\tau) = \sum_n \alpha_n \tau^n$ about $\tau = 0$ [see (2.8)] and evaluated at φ^2 corresponds precisely to the expansion of I in eigenfunctions of m. Hence, $\lim_{j\to\infty} m^j I = \alpha_0 = t(0) = \varphi_0^2$.

A similar reasoning can be applied to compute the action of m^j on other f's, leading to other correlations when $N = \infty$.

ACKNOWLEDGMENT

This work was partially supported by CNPq.

REFERENCES

[FOS] P. A. Faria da Veiga, M. O'Carroll, and R. Schor, A classical large N hierarchical vector model in three dimensions: A nonzero fixed point and canonical decay of correlation functions, *J. Math. Phys.*, to appear in March 1998.

[GK1] K. Gawedzki and A. Kupiainen, Non-Gaussian fixed points of the block spin transformation. Hierarchical model approximation, *Commun. Math. Phys.* **89**:191–220 (1983).

[GK2] K. Gawedzki and K., A. Kupiainen, Non-Gaussian scaling limits. Hierarchical model approximation, *J. Stat. Phys.* **35**:267–284 (1984).

[GK3] K. Gawedzki and A. Kupiainen, Asymptotic freedom and beyond perturbation theory, in *Phenomènes Critiques, Théories de Jauge et Systèmes Aléatoires*, K. Osterwalder and R. Stora, eds. Les Houches Summer School, 1984, Session (Elsevier, Amsterdam, 1986).

[GK4] K. Gawedzki and A. Kupiainen, Rigorous renormalization group and asymptotic freedom, in *Scaling and Self-Similarity in Physics*, J. Fröhlich, ed. (Birkhäuser, Boston, 1983).

[H] E. Hille, *Analytic Function Theory*, 2 vols. (Ginn and Company, Boston, 1962).

[N1] S. K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, New York, 1976). See also The renormalization group and the large n limit, *J. Math. Phys.* **15**:1866–1891 (1974).

[SO1] R. Schor and M. O'Carroll, Correlation functions and the Goldstone picture for the hierarchical classical vector model at low temperatures in three or more dimensions, *J. Stat. Phys.* **64**:163–191 (1991).

[SO2] R. Schor and M. O'Carroll, A low temperature properties of the hierarchical classical vector model, *Commun. Math. Phys.* **138**:487–505 (1991).

[SO3] R. Schor and M. O'Carroll, On the mass spectrum of the $2+1$ Gauge-Higgs lattice quantum field theory, *Commun. Math. Phys.* **103**:569–597 (1986).

[Z] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, Oxford, 1993).